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DESIGN OF A FLEXIBLE-WALLED SUPERSONIC NOZZLE

A Thesis

Presented to

the Faculty of the Department of Engineering

University of Virginia

In Partial Fulfillment

of the Requirements for the Degree

Master of Aeronautical Engineering

by

Jack E. Marte

March 1951

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LIST OF SYMBOLS

x	distance in direction of tunnel centerline
y	distance normal to tunnel centerline
A	upstream end of subsonic section
B	downstream end of subsonic section; geometric minimum
C	upstream end of supersonic section; aerodynamic minimum
D	downstream end of expansion portion of supersonic section
P	a uniform load
M	cross section bending moment
I	moment of inertia of cross section with respect to a perpendicular axis through its center.
E	modulus of elasticity in tension
p	pressure
ρ	density
ρ_δ	free stream density
θ	nozzle expansion angle
ϕ	velocity potential
u	velocity component in x-direction
v	velocity component in y-direction
a	velocity of sound
$\bar{w}$	total velocity vector
U	free stream velocity
μ	coefficient of viscosity $\frac{\text{stress}}{\partial u/\partial y}$
δ	boundary layer thickness
δ^*	displacement thickness $\left[\int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta U}\right) dy\right]$

ϑ	momentum thickness	$\left[\int_0^\delta \frac{\rho u}{\rho_\delta U}\left(1 - \frac{u}{U}\right)dy\right]$
τ_0	skin friction coefficient	$\left[\left(\mu\frac{\partial u}{\partial y}\right)\right]_{y=0}$

CHAPTER I

PURPOSE OF THE STUDY

This paper presents a method for the design of a flexible-walled supersonic nozzle. It originated as a design problem in the construction of a series of supersonic nozzles for the 4-foot supersonic pressure tunnel at Langley Field, Virginia.

The problem as presented was of a dual nature. It was required to design a supersonic nozzle which would have smooth flow at a given Mach number in the test section. Also, it was specified that the members which carry the load required to bend the flexible walls to form the nozzle should be uniformly stressed.

The area of the minimum section forthe design Mach number was known and the mechanical design of the mechanism for bending the flexible walls was complete so the locations of the applied loads were known. In addition, the locations of the various parts of the nozzle and its overall length were, to some extent, fixed by the physical design of the plant.

In the design of previous nozzles, which had been limited to the low Mach number range, structural considerations had a secondary role to the aerodynamic requirements; because little bending of the flexible plate was required to produce the design area ratio between the first minimum and the test section.

For high Mach numbers, however, where more severe bending of the plate is required, there are imposed serious structural limitations.

It is the purpose of this paper to show that under these conditions the structural and aerodynamic requirements can be satisfied jointly in a design of a practical nature.

CHAPTER II

VALIDATION OF THE IMPORTANCE OF THE PROBLEM

I. AERODYNAMIC RELATIONSHIPS

Consideration of the relationships between density, velocity, and area changes for isentropic flow in a channel which are obtainable from the equation of motion and the continuity equation reveals that for subsonic flows the speed increases more rapidly than the corresponding decrease in density so that to increase the speed in this region it is necessary to decrease the area of the channel in order to satisfy continuity considerations; on the other hand, when the flow is supersonic the speed increases less rapidly than the density decreases so that here the area must increase to increase the flow speed. Between these two regions is a point where the flow reaches the speed of sound and here the density change is such that no area change at all is required for an increment of speed. Thus, to generate a supersonic flow at a given Mach number it becomes necessary to design a channel which decreases in area to a minimum section and then increases until the design Mach number is reached.

In small supersonic wind tunnels, the usual practice is to use contoured blocks which produce the necessary area variation. A separate set of blocks is required for each Mach number at which the tunnel is to run. In a large supersonic wind tunnel this method becomes impractical because of the size, weight, and cost of fabrication of the nozzle blocks. The 4-foot supersonic pressure tunnel uses two flexible walls constructed from 0.55 inch stainless steel to accomplish this purpose.

II. STRUCTURAL DETAILS

The tunnel has a rectangular nozzle and test section, figure 1 , consisting of two fixed parallel side walls and the two horizontal flexible nozzle walls; the side walls and nozzle walls are 25 feet long and are continuous from a point 66 inches upstream of the throat to the end of the test section (figure 2). The test section has a width of 4.5 feet and a height which varies with the Mach number but which is 4.4 feet for the Mach 1.59 nozzle. The length of the uniform flow region along the wall of approximately 7 feet.

The supersonic nozzle and test section are formed by deflecting the horizontal flexible walls against a series of fixed interchangeable templates which are designed to produce uniform flow in the test section. The deflection of the nozzle walls is accomplished by means of jacking screws attached to transverse corrugations on the outside of the flexible walls. These corrugations, which are fastened to the flexible walls by means of studs welded to the under side of the plate, serve to increase the transverse stiffness of the nozzle plate and to distribute the jacking loads, thereby minimizing local wall irregularities. Details of this arrangement are shown in figure 1(b).

III. NATURE OF THE STRUCTURAL LOADS

In a design of this nature, it is necessary to consider the stresses arising from a number of causes including bending in the flexible plates, deflection due to pressure differences across the plates, and loads on the jacks and studs. Most of these sources of stress could be reduced by lengthening the nozzle but this would cause increased aerodynamic losses due to fricition at the walls and also the physical design of the existing tunnel configuration limits the length.

In an earlier nozzle designed for a Mach number of 1.59, some failures of the studs used to attach the corrugation to the flexible plate were experienced in the subsonic portion.

It is these studs which carry the loads applied by the jacks to cause the flexible walls to follow the contour templates. A stress analysis revealed that the loads carried by various studs were very unequal as a result of the distribution of bending stresses in the flexible walls which had been faired arbitrarily in this region. Since this condition would tend to become more serious with increasing Mach number it was decided that in the design of a Mach 2.2 nozzle an attempt should be made to evenly distribute the loads over all the studs, not only in the subsonic portion of the nozzle, but also in as much of the supersonic part as possible. This was to be accomplished by fairing the flexible walls in a curve which would require uniform loads at the support points to produce it. This curve of uniform load can be considered as a criteria which must be satisfied jointly with the usual aerodynamic requirements in this problem.

CHAPTER III

METHODS OF PROCEDURE

For each Mach number at which the tunnel is operated the nozzle templates must be computed for a section from the upstream end of the flexible wall to the end of the test section, a total distance of 25 feet. For general reference, a list of key station locations is given in table I and shown graphically in figure 2.

TABLE I

Point (figure 2)	Station (feet)	Description
	0	upstream flange of entrance cone
	11.425	upstream end of flexible wall
A	12.445	upstream end of region of uniform load
B	16.525	geometric minimum
C	17.294	aerodynamic minimum
D	21.374	downstream end of region of uniform load
	28.835	upstream end of test section
	36.0	downstream end of test section

In general the aerodynamic design procedures of this nozzle followed those used in earlier nozzles as concerned the application of the method of characteristics and the boundary layer corrections.

These nozzles have been built and tested and found to give thoroughly satisfactory results.

The method of procedure in this paper is to develop the end conditions consistent with the imposed aerodynamic and structural requirements and then apply them to the differential equation resulting from the elastic curve equation to produce an expression for the coordinates of the nozzle. This procedure is carried out seperately for the subsonic and supersonic portions of the nozzle. After the nozzle coordinates for potential flow have been obtained a correction is made for boundary layer growth and the region at the minimum section is adjusted to assure choking at the aerodynamic minimum.

CHAPTER IV

DESIGN OF SUBSONIC PORTION OF NOZZLE

It is possible to consider the subsonic portion of the nozzle independently even though the entire nozzle, both subsonic and supersonic parts, is bent from a continuous sheet of steel, because the summations of forces and moments at the downstream end of this portion (point B) of the nozzle are zero. The length in terms of jack spacing and the direction of the individual jack forces was chosen to bring about this result.

I. AERODYNAMIC REQUIREMENTS

In this portion of the nozzle, section AB in figure 2, the only aerodynamic requirements are that the flow be smooth and uniform at the minimum section and in addition, the nozzle shape must be continuous through the minimum.

II. THE EQUATION OF UNIFORM LOAD

The curve which results in uniform jack loads is obtained through the use of the well known equation of the elastic curve,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \tag{1}$$

Taking moments about B, figure 2,

$$M = P\,f(x)$$
$$= P\left[\left\{x + (x - 7)\right\} - \left\{(x - 14) + (x - 21) + (x - 28) + (x - 35)\right\} + \left\{(x - 42) + (x - 49)\right\}\right]$$

where clockwise rotation is taken to be positive.

P is one of the uniform loads; M is the bending moment at the cross section in question; and I is the moment of inertia of the cross section with respect to a perpendicular axis through its center. Integration with respect to x of equation (1) results in

$$\frac{dy}{dx} = \frac{P}{EI}\int f(x)dx + C_1, \tag{2}$$

an equation for the slope of the uniform load curve, where

$$f(x)dx = \frac{1}{2}\left[\left\{x^2 + (x - 7)^2\right\} - \left\{(x - 14)^2 + (x - 21)^2 + (x - 28)^2 + (x - 35)^2\right\} + \left\{(x - 42)^2 + (x - 49)^2\right\}\right]$$

A second integration with respect to x results in an equation for deflection which is the required curve of uniform load

$$y = \frac{P}{EI}\iint f(x)dx + C_1x + C_2 \tag{3}$$

where

$$\iint f(x)\,dx = \frac{1}{6}\left[\left\{x^3 + (x-7)^3\right\} - \left\{(x-14)^3 + (x-21)^3 + (x-28)^3 + (x-35)^3\right\} + \left\{(x-42)^3 + (x-49)^3\right\}\right]$$

Equation (1) is an equation of third order and requires three boundary conditions for a specific solution. These conditions are supplied by the requirements that the slope and ordinate of the flexible wall at the upstream end of the section must equal those of the flapper and at the downstream end be equal to the slope and ordinate of the minimum section. Since the flapper plate is inflexible and hinged at one end as shown in figure 2, its ordinate and slope are not independent so together they account for one boundary condition. The slope and ordinate at the downstream end of this section are independent variables and account for the other two boundary conditions.

The ordinate at the minimum section had previously been determined from the mass flow characteristics of the compressor involved as 15.09 inches. The condition that the section is a minimum requires that the slope must be zero there. Knowing these two end conditions, an expression involving the slope and ordinate of the curve at point A, figure 2, can now be obtained using the x , y axis system which has its origin at B[1].

---

[1]It should be noted here that the use of the three axis systems located as shown in figure 2 simplifies the application of the boundary condition and the expression of the differential equations.

At point B:

From equation (2)

$$\left(\frac{dy}{dx}\right)_B = \frac{P}{EI}\left[\int f(x)\,dx\right]_B + C_1$$

$$0 = 0 + C_1$$

so

$$C_1 = 0$$

From equation (3)

$$y_B = \frac{P}{EI}\left[\iint f(x)\,dx\right]_B + C_1 x + C_2$$

$$y_B = 0 + 0 + C_2$$

$$y_B = C_2$$

Now, for any point on the curve, equations (2) and (3) can be written as

$$\frac{dy}{dx} = \frac{P}{EI}\int f(x)\,dx \qquad (2\text{-a})$$

$$\frac{P}{EI} = \frac{dy}{dx}\Big/\int f(x)\,dx$$

and

$$y = \frac{P}{EI}\iint f(x)\,dx + y_B \qquad (3\text{-a})$$

$$\frac{P}{EI} = \frac{y - y_B}{\iint f(x)\,dx}$$

so that

$$\frac{dy/dx}{\int f(x)\,dx} = \frac{y - y_B}{\iint f(x)\,dx}$$

At point A, (x = 49),

$$\left(\frac{dy}{dx}\right)_A = \frac{\int f(x)\,dx}{\iint f(x)\,dx}\,(y_A - y_B)$$

$$\left(\frac{dy}{dx}\right)_A = \frac{784}{19208}(y_A - y_B)$$

$$\therefore \doteq \frac{y_A - 15.09}{24.5}$$

This expression defines the slope as a linear function of the ordinate for the elastic curve equation and is plotted in figure 3. A second curve of ordinate versus slope of the flapper plate end is plotted on the same figure using measurements obtained on a scale model of the flapper and their intersection provides a common solution which satisfies both the geometry of the flapper plate and the elastic curve equation. As obtained from figure 3, the values of slope and ordinate are $\left(\frac{dy}{dx}\right)_A = -0.1318$ and $y_A = 18.32"$.

Exact agreement in slope at point A is not critical since the flow is moving with low velocity at that point so the accuracy of the graphical solution is sufficient here.

Now shifting to the x,y axis system with origin at A, the constants of equation (3) can be evaluated and the equation of the curve written. From equation (2), evaluated at point B,

$$\left(\frac{dy}{dx}\right)_B = \frac{P}{EI}\left[\int f(x)\,dx\right]_B + C_1$$

$$C_1 = -\frac{P}{EI}\left[\int f(x)\,dx\right]_B$$

$$C_1 = -784\,\frac{P}{EI}$$

since $(dy/dx)_B = 0$ is an end condition.

Using equation (3) at point A, (x = 0),

$$y_A = \frac{P}{EI}\left[\iint f(x)\,dx\right]_A + C_1 x + C_2 \qquad (7)$$

$$C_2 = 18.32$$

since $y_A = 18.32''$ is a second end condition.

From equation (2) at point A, (x = 0),

$$\left(\frac{dy}{dx}\right)_A = \frac{P}{EI}\left[\int f(x)\,dx\right]_A + C_1$$

$$= C_1 = -\ .1318$$

since the third end condition is $(dy/dx)_A = -0.1318$.

Thus from equations (5) and (7)

$$\frac{P}{EI} = -\frac{C_1}{781}$$

$$= +\frac{.1318}{784} = .0001681$$

$$\frac{1}{6}\frac{P}{EI} = .00002802$$

Now the exact solution of the elastic curve equation which satisfies the boundary conditions can be written from equation (3)

$$y = .00002802\left[\left\{x^3 + (x-7)^3\right\} - \left\{(x-14)^3 + (x-21)^3 + (x-28)^3 + (x-35)^3\right\} + \left\{(x-42)^3 - (x-49)^3\right\}\right] - .1318x + 18.32 \qquad (8)$$

Solution of this equation for successive values of x gives the cordinate for the curve from the flapper plate to the geometric minimum. This is a general form of the equation which is true only if quantities in which $(x - 7n) < 0$ are neglected. A parallel wall extends one jack length (7 inches) downstream of B, the geometric minimum, to point C where the expansion portion of the supersonic diffuser begins. This configuration eliminates bending moments at B and C and also tends to produce a smooth flow over this critical region.

CHAPTER V

DESIGN OF SUPERSONIC PORTION OF NOZZLE

I. AERODYNAMIC REQUIREMENTS

The expansion portion of the supersonic effuser, (CD), was also designed as a curve which would result in uniform jack loads since this part of the effuser can be of arbitrary shape in the characteristic method used for the aerodynamic design of this portion of the nozzle. However, the length of the nozzle is fixed by the shape of the expansion part for a given Mach number. The elastic curve equation, equation (1), can again be used here and two of the required three boundary conditions are supplied by the slope and ordinate at C which are the same as those at B. The third condition is more difficult to obtain for the slope, or ordinate since they are functions of each other, at the point of maximum expansion, point D, determines the test section Mach number, other conditions being equal. This relationship between maximum expansion angle and Mach number can be obtained only by completing a characteristic net for the nozzle.

Since the test section Mach number is a design condition it is necessary to use successive approximations of the slope to obtain the required test section Mach number. In the design of this nozzle the third approximation resulted in a nozzle of the required Mach number to a sufficient degree of accuracy. The first two approximations were checked manually using a Mach net with two degree increments. The final calculation was made on a automatic relay computing machine for a Mach net having three-quarter degree increments using the standard setup of the machine for nozzle calculations.

Downstream of the point of maximum expansion, D, the shape of the nozzle is dictated by aerodynamic considerations since the method of characteristics uses this portion of the nozzle to cancel the expansion lines originating in the section CD so no uniform load curve is possible downstream of point D. However, this region is not expected to be heavily loaded for stress analysis of an earlier nozzle showed the high stress concentrations only in the subsonic part of the nozzle.

II. UNIFORM LOAD CURVE

After the slope of point D has been chosen the uniform load curve between C and D can be obtained from the elastic load curve, equation (1), in the same manner that it was obtained over the length AB.

Let $\frac{1}{2}\theta_{max} = 13.87^{\circ}$, where θ is the total angle of expansion. The end conditions can now be given as:

1. $y_C = y_B = 15.09''$
2. $\tan \theta_C = 0$
3. $\tan \theta_D = -0.2469$

For this calculation the origin of the axis system x_2, y_2 is taken at D. There equation 2 becomes, since x = C,

$$\left(\frac{dy}{dx}\right)_D = \frac{P}{EI} \int f(x)dx + C_1 \qquad (9)$$

$$-0.2469 = 0 + C_1$$

$$C_1 = -0.2469$$

At point C x = 49" and equation (2) is

$$\left(\frac{dy}{dx}\right)_C = \frac{P}{EI} \int f(x)\,dx + C_1 \qquad (10)$$

$$0 = \frac{P}{EI}(784) - 0.2469$$

$$\frac{P}{EI} \doteq \frac{.2469}{784} = .00315 \quad ; \quad \frac{1}{6}\frac{P}{EI} = .0000525$$

Equation (3) at point C is

$$y_c = \frac{1}{6}\frac{P}{EI} \iint F(x)\,dx + C_1 x + C_2 \qquad (11)$$

$$C_2 = 15.093 + (.2469)(49) - (.000315)(19208)$$

$$C_2 = 21.140$$

The equation for the curve CD can be written as

$$y = .0000525 \left[\left\{ x^3 + (x-7)^3 \right\} - \left\{ (x-14)^3 + (x-21)^3 - (x-28)^3 + (x-35)^3 \right\} + \left\{ (x-42)^3 + (x-49)^3 \right\} \right] - .2469x + 21.140 \qquad (12)$$

Here again, terms where $(x - 7n) < 0$ are to be neglected.

Successive solutions at various x-values result in a series of y-values from which a smooth curve, CD, can be plotted. Since the characteristic method of nozzle design requires that expansions be concentrated into finite steps it is necessary to approximate the smooth curve by a series of straight line segements. This is done for the final approximation by plotting the first derivative of the curve against the axial length from which coordinates at three-quarter degree increments can be obtained. Tangent lines were drawn at these points on the uniform load curve and their intersections were taken as the coordinates of the straight line segement approximation to the uniform load curve.

CHAPTER VI

THE METHOD OF CHARACTERISTICS

I. DERIVATION OF CHARACTERISTIC EQUATIONS IN SUPERSONIC FLOW

This use of characteristics has been in general practice in supersonic aerodynamics for some time and a mathematical development of the characteristic equations can be found in references such as Liepmann and Puckett[2], and Courant and Friedrichs[3] which present such a development with varying degree of vigour.

On a physical plane the characteristic equations for two dimensional steady perfect isentropic flow can be developed from the continuity equation and the equations of motion which are respectively

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{13}$$

and

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned} \tag{14}$$

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[2]Hans Wolfgang Liepmann and Allen E. Puckett, Introduction to Aerodynamics of a Compressible Fluid (New York: John Wiley and Sons, 1947), p 210.

[3]R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves (New York: Interscience Publishers, Inc. 1948) p 40.

where u and v are velocity components in the x and y-directions, p is pressure, and ρ is density. Since the velocity of sound can be written as

$$a = \left(\frac{\partial p}{\partial \rho}\right)^{\frac{1}{2}} \tag{13}$$

and

$$\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial x} = \frac{1}{a^2}\frac{\partial p}{\partial x} \tag{14}$$

the equations of continuity and motion can be combined to result in

$$\frac{\partial u}{\partial x}\left(1-\frac{u^2}{a^2}\right)+\frac{\partial v}{\partial y}\left(1-\frac{v^2}{a^2}\right)-\frac{uv}{a^2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=0 \tag{15}$$

Isentropic flow requires irrotationality which is expressed as

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

therefore a velocity potential ϕ defined by the conditions

$$\frac{\partial \phi}{\partial x} = u \quad \text{and} \quad \frac{\partial \phi}{\partial y} = v$$

exists. On substitution of the velocity potential and the condition of irrotationality equation (15) becomes the potential equation of motion for compressible flow in two dimensions.

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2 \phi}{\partial x^2} - \frac{2uv}{a^2}\frac{\partial^2 \phi}{\partial x \partial y} + \left(1-\frac{v^2}{a^2}\right)\frac{\partial^2 \phi}{\partial y^2} = 0 \tag{16}$$

This equation can be rewritten as

$$Hr + 2Ks + Lt = 0 \tag{17}$$

where

$$r = \frac{\partial^2 \phi}{\partial x^2} \quad ; \quad s = \frac{\partial^2 \phi}{\partial x \partial y} \quad ; \quad t = \frac{\partial^2 \phi}{\partial y^2}$$

$$H = 1 - \frac{u^2}{a^2} \quad ; \quad K = -\frac{uv}{a^2} \quad ; \quad L = 1 - \frac{v^2}{a^2}$$

Physical significance can be attached to characteristics by considering a supersonic flow at a Mach line. Here the magnitude and direction of the flow upstream of the Mach line are not identical to the values of flow magnitude and direction which exist downstream. In terms of equation (14); if velocity potential ϕ, exists upstream of the Mach line then a new potential, ϕ_2, will exist on the downstream side. Since the flow is continuous over a Mach line the potentials and their first derivatives which are the velocity components must be equal on the line. The Cauchy theorem[4] shows that the integral of a partial differential equation such as equation (16) is universally defined if its value and the value of its first derivatives are known for all points on a line. This does not occur only if the line is a characteristic variety of the equation. Since the two solutions of the potential flow equation, equation (16), for the regions upstream and downstream

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[4]C. A. Webster, Partial Differential Equations of Mathematical Physics (New York; G. E. Stechert Co., 1933), Chapter VI.

of the Mach line have the same value of potential and the same derivatives at the Mach line but not elsewhere it can be concluded that the Mach lines in a supersonic flow coincide with the characteristic lines of the potential flow equation. This means that the mathematical properties of characteristics, discussed in detail in Courant and Hilbert,[5] can be assigned to Mach lines and makes possible a step-by-step solution of the potential flow equation in a supersonic field.

Now consider some curve C in the flow and let λ be an arc length along the curve. Thus ϕ, u, and v are functions of λ. Then

$$\frac{\partial u}{\partial \lambda} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \lambda} = r\frac{\partial x}{\partial \lambda} + s\frac{\partial y}{\partial \lambda} \tag{18}$$

$$\frac{\partial v}{\partial \lambda} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \lambda} = s\frac{\partial x}{\partial \lambda} + t\frac{\partial y}{\partial \lambda} \tag{19}$$

Since the coordinates of the curve are assumed known, r, s, and t are the three unknowns and equations (18) and (19) plus the equation of motion, (17), provide the necessary three equations to solve for the unknowns.

---

[5] R. Courant and D. Hilbert, <u>Methoden der Mathematischen Physik</u> (Berlin: Springer, 1937), vol II p 290.

$$
\begin{aligned}
r \frac{\partial x}{\partial \lambda} + s \frac{\partial y}{\partial \lambda} \qquad\qquad - \frac{\partial u}{\partial \lambda} &= 0 \\
s \frac{\partial x}{\partial \lambda} + t \frac{\partial y}{\partial \lambda} - \frac{\partial v}{\partial \lambda} &= 0 \\
rH + s(2K) + tL \qquad\qquad &= 0
\end{aligned}
$$

Using determinates

$$
s = \frac{\begin{vmatrix} \frac{\partial x}{\partial \lambda} & 0 & \frac{\partial u}{\partial \lambda} \\ 0 & \frac{\partial y}{\partial \lambda} & \frac{\partial v}{\partial \lambda} \\ H & L & 0 \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} & 0 \\ 0 & \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} \\ H & 2K & L \end{vmatrix}} \tag{20}
$$

Consider the case where S is indeterminant because both its numerator and denominator equal zero,

$$
\begin{vmatrix} x' & 0 & u' \\ 0 & y' & v' \\ H & L & 0 \end{vmatrix} = 0 \tag{21}
$$

and

$$
\begin{vmatrix} x' & y' & 0 \\ 0 & x' & y' \\ H & 2K & L \end{vmatrix} = 0 \tag{22}
$$

where $(\ )' = \frac{\partial}{\partial \lambda}(\ )$

Equation (22) expands to

$$x'\left[L x' - 2K y'\right] + y'\left[H y'\right] = 0,$$

or

$$L(x')^2 - 2K x'y' + H(y')^2 = 0$$

Division by $(x')^2$ gives

$$H\left(\frac{dy}{dx}\right)^2 - 2K\left(\frac{dy}{dx}\right) + L = 0 \qquad (23)$$

which has the solutions

$$\frac{dy}{dx} = \frac{K \pm \sqrt{K^2 - HL}}{H} \qquad (24)$$

If it is supposed that u and v are known at every point, the two directions given by equation (24) are defined and two families of curves may be located in the flow field. These are called characteristic curves. In terms of velocities equation (24) becomes

$$\frac{dy}{dx} = \frac{-\frac{uv}{a^2} \pm \sqrt{\frac{u^2v^2}{a^4} - \left(1 - \frac{u^2}{a^2}\right)\left(1 - \frac{v^2}{a^2}\right)}}{1 - \frac{u^2}{a^2}}$$

$$= \frac{-\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2} - 1}}{1 - \frac{u^2}{a^2}}$$

Thus the characteristics are real if $q^2 = u^2+v^2 \geq a^2$ and imaginary if $q^2 < a^2$. This corresponds to the supersonic and subsonic conditions, respectively.

The relationship obtained by setting the numerator of S, equation (21), equal to zero gives

$$x'\left[-Lv'\right] + u'\left[-Hy'\right] = 0,$$

or

$$H\frac{\partial u}{\partial \lambda}\frac{\partial y}{\partial \lambda} + L\frac{\partial x}{\partial \lambda}\frac{\partial v}{\partial \lambda} = 0. \qquad (25)$$

This equation gives the variation of u and v along the characteristic lines.

II. METHOD OF CHARACTERISTICS IN SUPERSONIC NOZZLE DESIGN

In the application of the characteristic method to supersonic nozzle design used here uniform sonic flow is assumed at the aerodynamic minimum. The walls of the nozzle must diverge after the throat in a manner which in general is arbitrary but which in this case is prescribed by the uniform load condition. This is the expansion portion of the nozzle, which if it treated as a series of straight line segements with a finite increment of slope between each, will produce a flow pattern of intersecting expansion waves as shown in figure 4.

At some point the curvature must reverse until the walls again become parallel. If the curve in this region is also considered to be made up of straight line segements, the corners are now concave and compression waves are produced. If these corners are located at the points where the expansion waves strike the wall and if the compression waves originating from those corners are of identical strength to the incoming waves then they cancel each other and no disturbance is reflected or produced. When all the expansion waves have thus been canceled the walls will be parallel and the flow will be undisturbed and therefore parallel to the walls throughout the region at some Mach number greater than one.

The length of the nozzle for a given Mach number is a function of the maximum expansion angle and the number of reflections of the expansion waves before cancellation. In this case the length of the nozzle is set indirectly by the uniform load structural condition. Since this nozzle is to be used in an existing tunnel configuration it is necessary that the nozzle length fall within a given range of values. The final nozzle configuration obtained from the sucessive approximations fell within this range so it was possible to satisfy all the imposed conditions. In general the nozzle length would not be restricted as it was in this particular problem.

CHAPTER VII

BOUNDARY LAYER CORRECTION

I. INTRODUCTION

Up to this point all of the aerodynamic calculations have dealt with potential flow, i.e., the effects of viscosity were neglected. In flows of this type the only boundary condition at the wall is that the direction of the flow must be tangent to the wall. The flow is free to slip along the surface. However, theory and experiment have shown that the interaction of forces between molecules in the flow and in the wall is so strong in flows of normal density that the fluid particles adjacent to the wall will have the same velocity as the wall itself. This condition is incompatible with the requirements for potential flow. The fluid at some distance from a wall has a finite velocity relative to the wall while at the surface of the wall the relative velocity is zero. This decrease from finite to zero relative velocity is brought about by internal friction of viscosity.

II. DERIVATION OF BOUNDARY LAYER MOMENTUM EQUATIONS

Consideration of viscosity introduces a new boundary condition which means that instead of the first order Eulerian equations of motion, (14), it is now necessary to deal with second order equations, the so-called Navier-Stokes equations. In two dimensional flow for air they can be written as

$$
\begin{aligned}
\rho \frac{Du}{Dt} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\operatorname{div}\bar{w}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
\rho \frac{Dv}{Dt} &= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\operatorname{div}\bar{w}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]
\end{aligned}
\tag{26}
$$

where μ is the coefficient of viscosity and

$$\operatorname{div}\bar{w} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

The terms of the left side of these equations and the first term on the right comprise Euler's equation of motion. The div $\bar{w}$ term is due to compressibility and the other terms result from the action of viscosity.

Experimental results show that even in a viscous fluid such as air the flow, except for a thin layer next to the surface of a body, acts almost as if the air were a non-viscous fluid. In this thin layer the velocity changes from its free stream value at the outer edge to zero at the edge adjacent to the wall. Since this change in velocity, which can be large, occurs in a thin layer, the rate of change of velocity normal to the flow direction $(\partial u/\partial y)$ can be very large. This viscous force per unit volume is $\tau_0 = \mu\frac{\partial u}{\partial y}$ so that in the layer this force can be appreciable even though μ, the coefficient of viscosity, is small for a medium such as air. Here the viscous and inertia terms are of the same order of magnitude. Outside of this layer the normal velocity gradient is small so the viscous forces are very small compared to the inertia terms and the simpler potential flow equations can be used in place of the Navier-Stokes equations.

Following this line of reasoning, Prandtl in 1904 was able to simplify the Navier-Stokes equations through an analysis of the order of magnitude of the terms so that a new set of equations of motion could be written for this region which he called the boundary layer. These equations are commanly known as the boundary layer equations and in two dimensions they are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad (27)$$

and

$$\frac{\partial p}{\partial y} = 0$$

In obtaining equations (27), the assumption was made that the variation of the velocity components with time were of the same order of magnitude as the rest of the acceleration terms.

Even though simpler than the Navier-Stokes equations the boundary layer equations are still of the second order and it is not generally possible to obtain solutions of these equations. An approximate solution can be obtained if the idea of attempting to satisfy the equations of motion for each individual particle is abandoned in favor of satisfying only the integral of all the particles over a section of the boundary layer. This amounts to satisfying the momentum equation which is obtained by direct integration of the boundary layer equations for two dimensional compressible flow as follows:

(28)

where δ is the boundary layer thickness.

Consider first

$$\int_0^\delta v \frac{\partial u}{\partial y} = \Big[uv\Big]_0^\delta - \int_0^\delta u \frac{\partial v}{\partial y}\, dy$$

and from continuity

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

so that

$$\int_0^\delta \frac{\partial v}{\partial y}\, dy = \Big[v\Big]_0^\delta = (v)_{y=\delta} = -\int_0^\delta \frac{\partial u}{\partial x}\, dy$$

Also

$$\Big[uv\Big]_0^\delta = U v_\delta = -U\int_0^\delta \frac{\partial u}{\partial x}\, dy, \quad \text{where } U = u_{y=\delta}$$

Therefore, the second and third terms of (19) become

$$\begin{aligned}\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)dy &= \int_0^\delta u\frac{\partial u}{\partial x}\, dy - U\int_0^\delta \frac{\partial u}{\partial x}\, dy + \int_0^\delta u\frac{\partial u}{\partial x}\, dy \\ &= \int_0^\delta 2u\frac{\partial u}{\partial x}\, dy - U\int_0^\delta \frac{\partial u}{\partial x}\, dy \\ &= \frac{\partial}{\partial x}\left\{\int_0^\delta u^2 dy - U\int_0^\delta u\, dy\right\}\end{aligned}$$

since the extra terms arising from the variability of the upper limit with x cancel out.

Also $$\int_0^\delta \frac{\partial p}{\partial x}\, dy = \delta\frac{\partial p}{\partial x}$$

and $$\int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy = \left[\frac{\partial u}{\partial y}\right]_0^\delta = -\left[\frac{\partial u}{\partial y}\right]_{y=0} = -\frac{\tau_0}{\mu}$$

since the skin friction coefficient, τ_0 is defined as

$$\tau_o = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

Hence the complete integration which is the momentum equation is

$$\tau_o = U \frac{\partial}{\partial x} \int_0^\delta \rho u \, dy - \frac{\partial}{\partial x} \int_0^\delta \rho u^2 dy - \int_0^\delta \frac{\partial u}{\partial t} dy - \delta \frac{\partial p}{\partial x} \qquad (29)$$

Using the equations of motion for a compressible fluid at the edge of the boundary layer, equation (13), the definitions:

$$\text{displacement thickness} = \delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_\delta U}\right) dy$$

and

$$\text{momentum thickness } \vartheta = \int_0^\delta \frac{\rho u}{\rho_\delta U} \left(1 - \frac{u}{U}\right) dy$$

and considering the flow to be steady the momentum equation can be rewritten as

$$\frac{\partial \vartheta}{\partial x} + \vartheta \left(\frac{H_c + 2}{U} \frac{dU}{dx} + \frac{1}{\rho_\delta} \frac{\partial \rho_\delta}{\partial x} \right) = \frac{\tau_o}{\rho_\delta U^2} \qquad (30)$$

where $H_c = \frac{\delta^*}{\vartheta}$.

This equation is applicable both to subsonic and supersonic flow but cannot be used across a shock wave because this condition violates the assumptions of the boundary layer equations.

The boundary layer along the nozzle walls was computed by the method of reference 6 which uses as a basis equation (30).

and it was assumed to originate at station 0, the upstream flange of the entrance cone.

III. METHOD OF CORRECTION FOR NOZZLE BOUNDARY LAYER

The method of reference 6 is to obtain an expression for momentum thickness in a turbulent boundary layer by a step-by-step integration process of equation (21) in which a fixed velocity profile is used. The effect of density changes through the boundary layer on the ratio of displacement thickness to momentum thickness is considered and the experimental result of Theodorsen and Regier[6] that skin friction coefficient for turbulent flow is independent of Mach number is used. The additional assumption is made that skin-friction formulas for flat plates may be used in the computation of boundary layer thickness in flow with pressure gradients.

Knowing the momentum thickness along the nozzle wall the displacement thickness is easily obtainable and it is this parameter which is used as the basis of the boundary layer correction.

Since the design of the tunnel fixes the vertical side walls as parallel, a rough method of correcting for the

[6]Theodore Theodorsen and Arthur Regier, Experiments on Drag of Revolving Disks, Cylinders, and Streamline Rods at High Speeds. NACA TR No. 793, 1944.

side wall boundary layer growth is used. The growth of boundary layer on the side walls is assumed identical to that on the top and bottom of the tunnel. Therefore, the boundary layer corrections to the top and bottom are determined by multiplying the previously calculated boundary layer displacement thicknesses by the ratio of the semi-perimeter of the nozzle to the nozzle width. These corrections are then applied solely to the top and bottom walls.

IV. ADJUSTMENT IN REGION OF AERODYNAMIC MINIMUM

In section AC (stations 16.525 to 17.294) in which the tunnel walls are parallel prior to correction for boundary layer growth the correction is intentionally made insufficient to allow for the full boundary layer growth so that choking will occur at point C. From C to the end of the nozzle, the coordinates from the characteristic calculation were opened an amount equivalent to the boundary layer displacement thickness to allow for boundary layer growth. These calculations are based on one atmosphere stagnation pressure in the test section.

A table of the nozzle coordinates with and without the boundary layer correction factor is presented in table II. The values with boundary layer correction can be used directly to lay out the templates for this nozzle.

TABLE II. ORDINATES OF 2.2 NOZZLE

SUBSONIC TEMPLATE

x (feet)	y (inches) (includes boundary layer correction)	y (inches) (magnitude of boundary layer correction)
11.025	20.40	0
11.425 (A)	19.80	0
12.025	18.91	0
12.525	18.17	0
12.695	17.928	0
12.945	17.537	0
13.195	17.157	0
13.445	16.792	0
13.695	16.454	0
13.945	16.149	0
14.195	15.881	0
14.445	15.655	0
14.695	15.471	0
14.945	15.330	0
15.195	15.228	0
15.445	15.160	0
15.695	15.122	0
15.945	15.102	0
16.195	15.098	0
16.445	15.094	0
16.525 (B)	15.093	0

SUPERSONIC TEMPLATE

x (feet)	y (inches) (includes boundary layer correction)	y (inches) (magnitude of boundary layer correction)
16.525 (B)	15.0930	0
16.9	15.1090	0.016
17.294 (C)	15.1191	.0231
17.730	15.1600	.0280
18.090	15.2080	.0280
18.282	15.2780	.0260
18.405	15.3365	.0245
18.515	15.3790	.0190
18.618	15.4634	.0194
18.721	15.5846	.0206
18.856	15.7318	.0238
18.987	15.8900	.0260
19.112	16.0598	.0278
19.235	16.245	.0330
19.361	16.453	.0370
19.487	16.674	.0420

x (feet)	y (inches) (includes boundary layer correction)	y (inches) (magnitude of boundary layer correction)
19.561	16.8205	0.0445
19.628	16.9550	.0470
19.747	18.2135	.0535
19.878	17.5320	.060
20.030	17.9130	.069
20.151	18.2324	.0764
20.278	18.5759	.0839
20.527	19.2780	.102
20.867	20.2519	.1279
21.262	21.3272	.1502
21.646	22.3064	.1904
22.032	23.2267	.2227
22.463	24.1880	.260
22.895	25.0790	.299
23.324	25.8969	.3369
23.760	26.6700	.3780
24.215	27.3850	.4210
24.673	28.0415	.4655
25.076	28.5680	.500
25.481	29.0250	.5490
25.972	29.4935	.5975
26.498	29.9184	.6504
27.069	30.2925	.7125
27.614	30.5660	.7700
28.184	30.7500	.8220
28.835 (E)	30.9090	.8850
30.00	31.0159	.9919
36 (F)	31.5400	1.516
38.0	31.7020	1.6780

(A) upstream end of flexible wall
(B) geometric minimum
(C) aerodynamic minimum
(E) beginning of test section
(F) downstream end of flexible wall

CHAPTER VIII

ACCURACY OF RESULTS

I. STRUCTURAL RESULTS

The structural requirement of uniform loads at the support points need only be approximately satisfied to prevent the original failures of the studs so no high degree of accuracy is necessary for the structural phase of the design.

II. AERODYNAMIC RESULTS

The aerodynamic design procedures followed those used in previous nozzles and the same degree of accuracy was obtained in each case. Since these earlier nozzles were satisfactory it is believed that the accuracy of this phase of the design for this nozzle is also sufficient.

The approximation of a boundary layer growth on the side walls which is identical to that on the top and bottom walls in computing the boundary layer correction is known to be inexact since the stream-wise pressure gradient is not the identical in each case and also the interaction effects in the corners of the rectangular tunnel are neglected. This approximation is used because of the difficulty attendant to obtaining a more exact solution and because previous experience has shown it to give satisfactory results in a similar application. Another source of error in the boundary layer calculations is the use of a constant stagnation pressure of one atmosphere although the nozzle is to be used over a range of stagnation pressures from 0.125 to 2.2 atmospheres. This approximation of constant stagnation pressure at the test section in the boundary layer calculations required if a single set of templates is to be used for each Mach number.

CHAPTER IX

RECOMMENDATIONS AND LIMITATIONS

In view of the roughness of the approximations involved in considering the boundary layer growth in this nozzle it is now considered likely that a less elaborate calculation of the boundary layer growth would reduce the computational effort simpler form of the momentum equation would result without decreasing the over-all accuracy of the design. It might even be possible to eliminate the boundary layer calculations entirely and design the nozzle by the method of characteristics for a slightly higher test section Mach number than is actually desired.

It has previously been noted that the designation of the uniform load curve in the expansion portion of the supersonic nozzle fixes the total length of the nozzle. This is an example of the type of limitation which may result indirectly when additional conditions are added to those resulting from the aerodynamic design. The situation in which the outside requirements are incompatible with the aerodynamic requirements can easily arise; in which case it will usually be necessary to defer the outside requirements to the aerodynamics since the function of the nozzle design will otherwise be impaired.

CHAPTER X

SUMMARY

The problem of the design of a nozzle to be used in a flexible-walled supersonic wind tunnel by the method of characteristics which will satisfy the structural requirement of uniform load on the flexible wall supports is carried out to a practical solution. Attention is given to the background of the method of characteristics and also to method of correcting the nozzle for boundary layer growth. The final results have been presented in table II in a form which gives nozzle coordinates and the magnitude of boundary layer correction.

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BIBLIOGRAPHY

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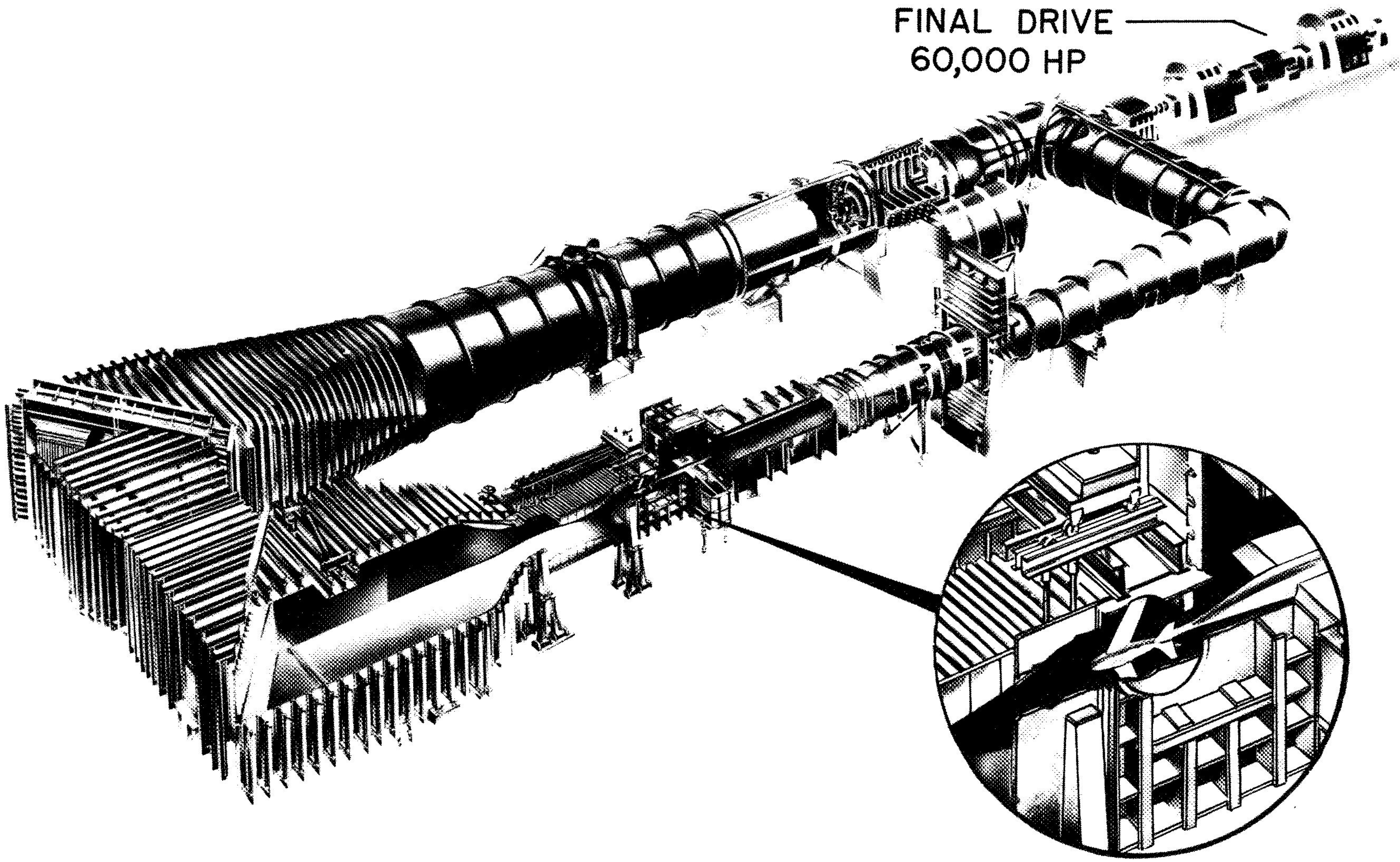

Figure 1.- 4- by 4-foot supersonic pressure tunnel.

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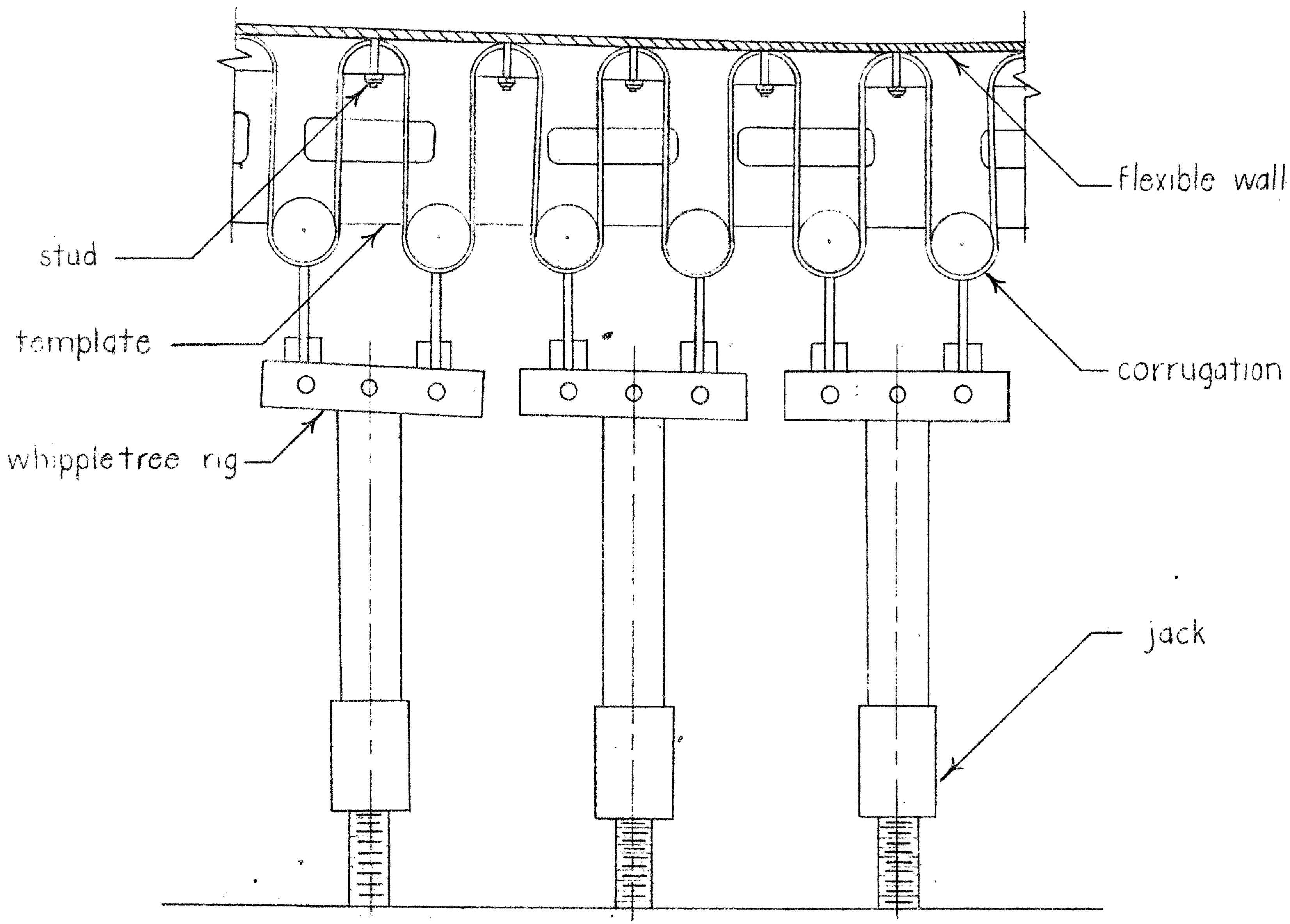

Figure 1 - concluded Section-view of flexible wall

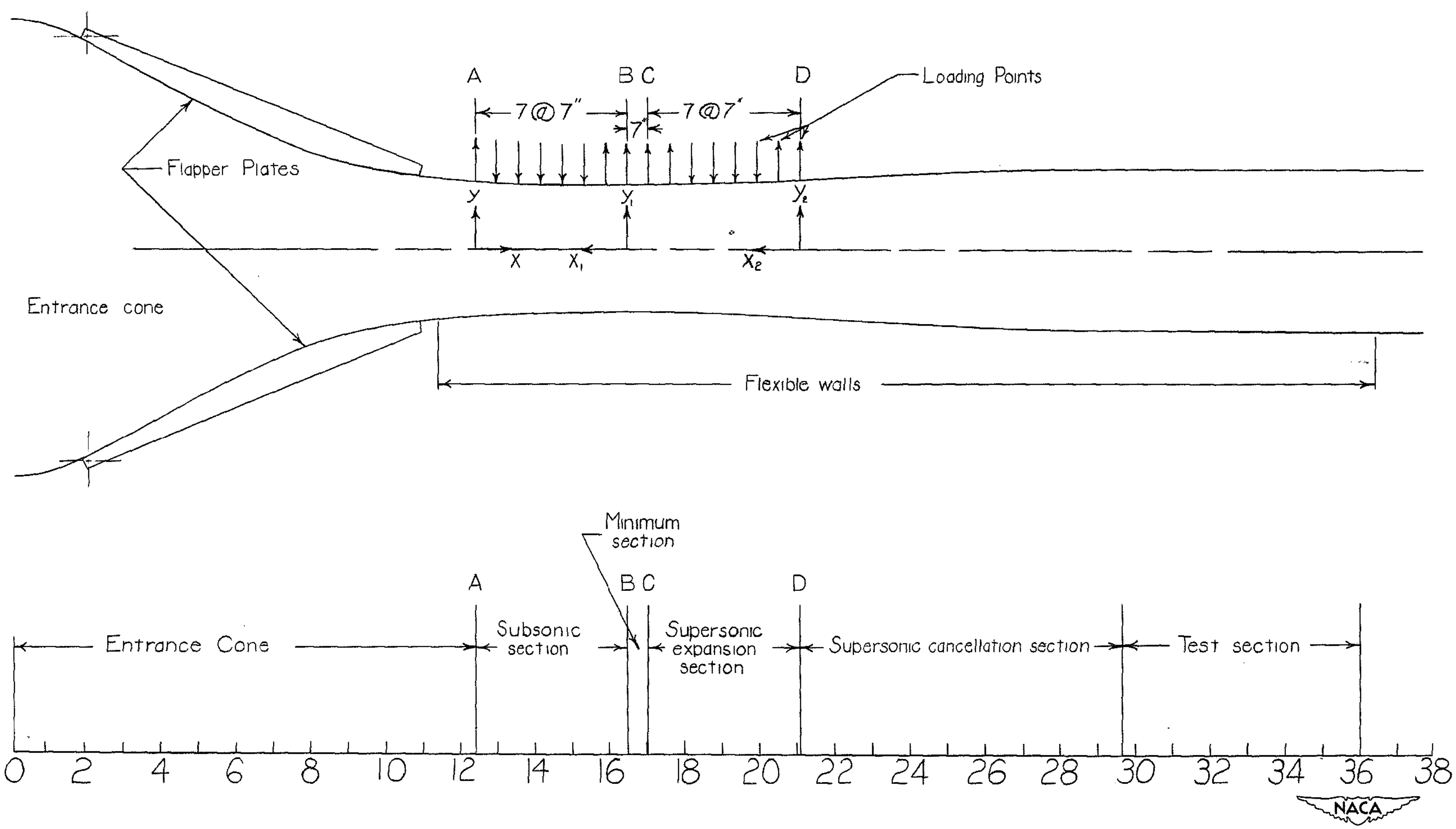

Figure 2.- Schematic layout of entrance cone, nozzle and test section.

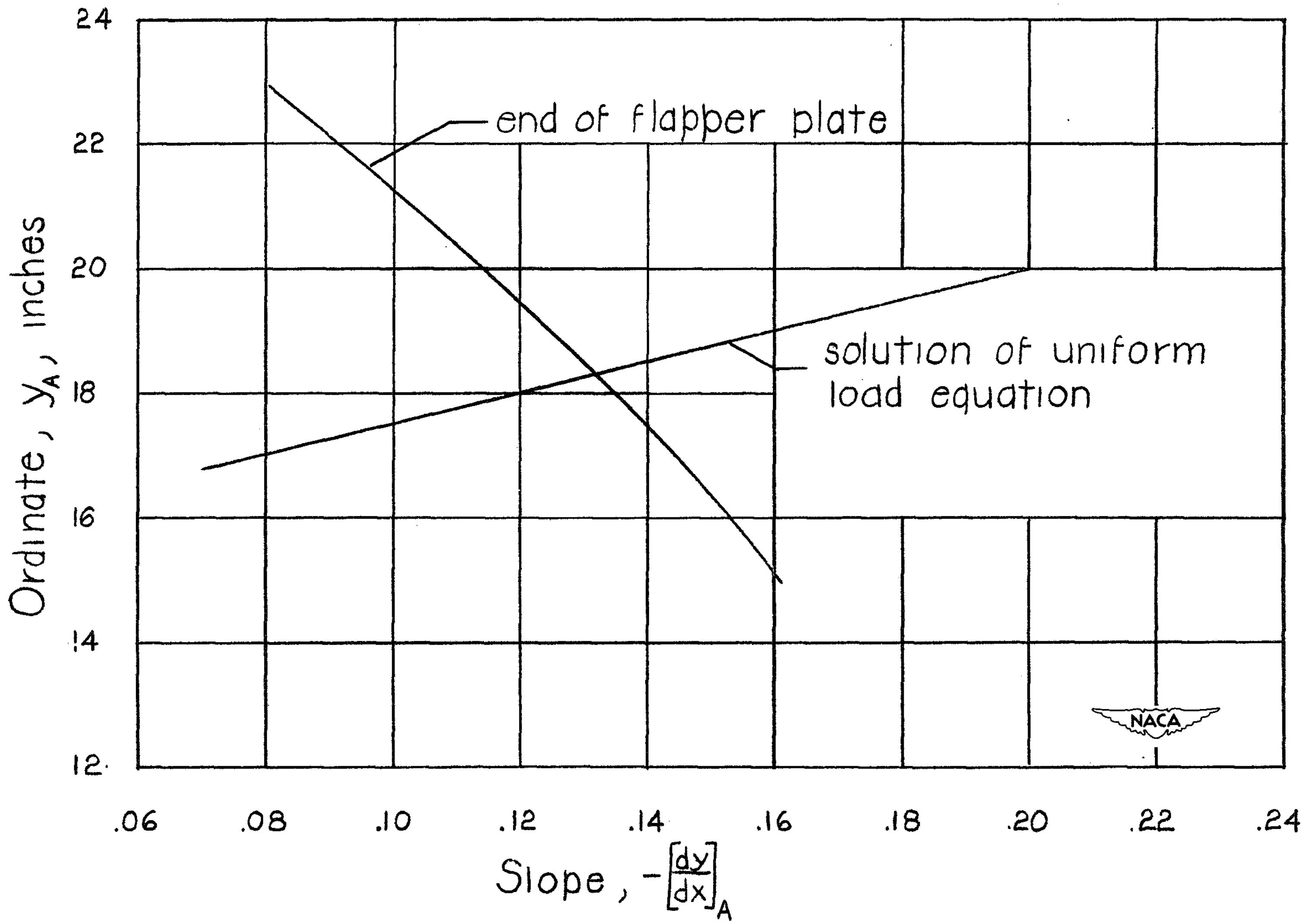

Figure 3.- Graphical solution for end conditions at point A.

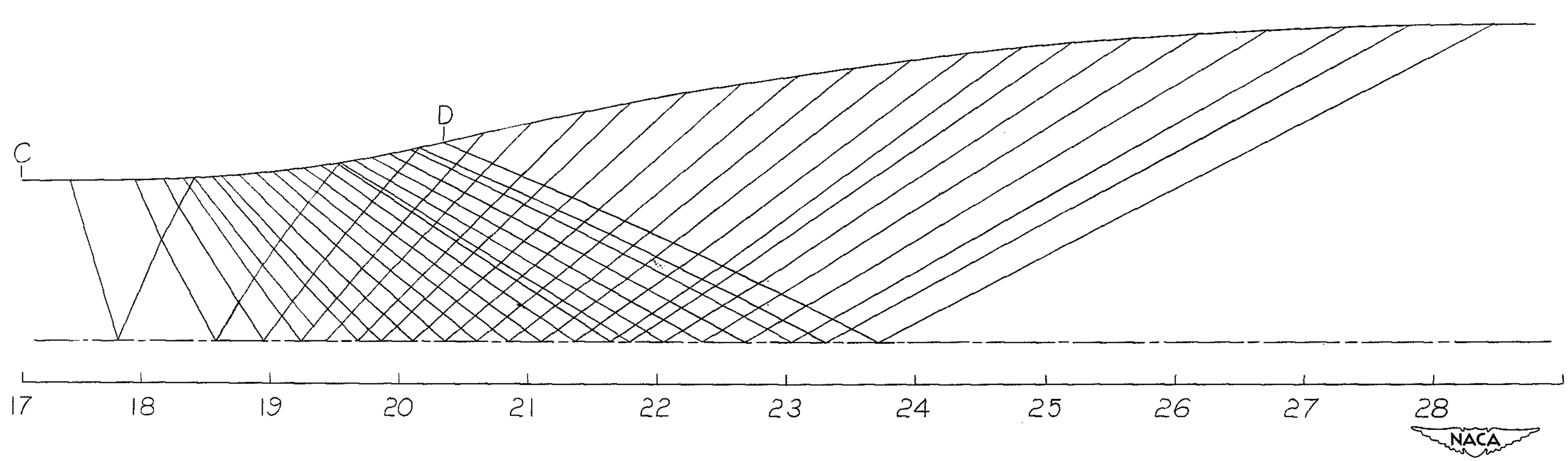

Figure 4.- Characteristic diagram.